

# MINIMAL ROOT'S EMBEDDINGS FOR GENERAL STARTING AND TARGET DISTRIBUTIONS

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November, 2016

## Abstract

Recent works (Dupire [2005], Cox and Wang [2013], Gassiat et al. [2015]) have studied the construction of Root's embedding. However, all the results so far rely on the assumption that the corresponding stopped process is uniformly integrable, which is equivalent to *the potential ordering condition*  $U_\mu \leq U_\nu$  when the underlying process is a local martingale.

In this paper, we study the existence, construction and optimality of Root's embeddings in the absence of the potential ordering condition. To this end, we replace the uniform integrability condition by the minimality condition (Monroe [1972a]). A sufficient and necessary condition (in terms of local time) for the minimality of embeddings is given. Using this result, we derive a constructive proof of the existence of a minimal Root embedding for general starting and target distributions (i.e.  $U_\mu \not\leq U_\nu$ ). We also discuss the optimality of such minimal embeddings. These results extend the generality of the construction method given by Cox and Wang [2013].

## 1 Introduction

Suppose that  $X = \{X_t\}_{t \geq 0}$  is a stochastic process on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ , and  $\nu$  and  $\mu$  are probability distributions on the state space of  $X$ , the Skorokhod embedding problem for  $(X, \nu, \mu)$  is:

*Given  $X_0 \sim \nu$ , to find a stopping time  $\tau$  s.t.  $X_\tau \sim \mu$ .* (SEP)

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\*The author is supported by the National Natural Science Foundation of China (NSFC No.11601306). This work was partially completed while the author visited Sapienza University of Rome as a postdoctoral fellow (SECS-S/06, 2013–2015). E-mail: [jiajie.wang@bath.edu](mailto:jiajie.wang@bath.edu).

This problem was initially proposed by Skorokhod [1965].

Under the classical setting where  $X$  is a Brownian motion starting at 0 and the target distribution  $\mu$  has zero mean, there is a rich literature regarding this problem, for example, Dubins [1968], Root [1969], Röst [1971], Monroe [1972b], Chacon and Walsh [1976], Azéma and Yor [1979], Vallois [1983], Chacon [1985], Perkins [1986], etc.. Many of the results above can be generalised to the cases where the underlying process is a martingale. We will not state them one by one in details. Instead, we refer curious readers to the survey paper Obłój [2004].

The results mentioned above focus on the cases where the embeddings are namely *UI stopping times*. Here, a stopping time  $\tau$  is a UI stopping time if the corresponding stopped process  $X^\tau := \{X_{t \wedge \tau}\}_{t \geq 0}$  is uniformly integrable, otherwise we call  $\tau$  a *non-UI stopping time*. When the underlying process is a continuous local martingale, Obłój [2004, Prop. 8.1] shows that there exists a UI embedding for  $\text{SEP}(X, \nu, \mu)$  if and only if

$$U_\mu(x) \leq U_\nu(x) < \infty, \quad \text{for all } x \in \mathbb{R}, \quad (\text{potential ordering})$$

where the function  $U_\mu$  is called *the potential of  $\mu$*  (Chacon [1977]):

$$U_\mu(x) := -\mathbb{E}^{Y \sim \mu}[|Y - x|] = -\int_{\mathbb{R}} |y - x| \mu(dy).$$

In this paper, we are concerned with  $\text{SEP}(X, \nu, \mu)$  in the absence of *potential ordering*. In such circumstances we cannot expect the corresponding embedding to be a UI stopping time.

For example, suppose that the initial distribution is the Dirac measure  $\nu = \delta_0$  and the target is  $\mu = \delta_1$ . The mean values of  $\nu$  and  $\mu$  do not agree, and *potential ordering* fails here. The hitting time  $H_1 = \inf\{t \geq 0 : W_t = 1\}$  is obviously an embedding for  $\text{SEP}(W, \nu, \mu)$  but obviously it is not a UI stopping time. Another example is that  $\nu = (\delta_1 + \delta_{-1})/2$  and  $\mu = \delta_0$ . The mean values agree, but *potential ordering* fails as  $U_\nu \leq U_\mu$ . The hitting time  $H_0 = \inf\{t \geq 0 : W_t = 0\}$  is a non-UI embedding for  $\text{SEP}(W, \nu, \mu)$ .

As presented above, when *potential ordering* fails, we cannot restrict attention to UI stopping times for embeddings. Instead, we may pose some other restrictions. For example, Pedersen and Peskir [2001] posed an integrability condition on the maximum of the scale function of  $X$  as the replacement of UI condition. After that, Cox and Hobson [2006] proposed another criterion on stopping times, which was initially introduced by Monroe [1972a]:

**DEFINITION 1.1** (Minimal stopping time). A stopping time  $\tau$  for the process  $X$  is minimal whenever  $\theta \leq \tau$  is a stopping time such that  $X_\theta$  and  $X_\tau$  have the same distribution, then  $\tau = \theta$ , a.s..

According to the definition, minimal stopping time could be a natural choice for "good" solutions of the embedding problem in a general context. For example, as

stated in [Hobson \[2011, Sec. 4.2\]](#), there exists a trivial solution for [SEP](#) in the general cases — simply run the process  $X$  until it firstly hits the mean of  $\mu$ , and thereafter can use any regular embedding mentioned above. The embeddings constructed in this way are always minimal stopping times, see [Cox and Hobson \[2006\]](#).

[Cox and Hobson \[2006\]](#) have made significant effort in the study of minimal stopping times for the Brownian motion starting at 0. A group of necessary and sufficient conditions for minimality is given. After that, [Cox \[2008\]](#) extended the previous results to the cases of general starting distributions. Thanks to these results, some well-known embeddings have been extended to the cases in which the [potential ordering](#) fails, for examples, the embeddings of the following types: Chacon-Walsh's, Azéma-Yor's, Vallois'.

In this work we are concerned with embeddings of Root's type which was initially proposed by [Root \[1969\]](#). Formally, suppose that  $W$  is a Brownian motion starting at zero and the target distribution is a centred distribution with finite second moment, [SEP](#)( $W, \delta_0, \mu$ ) admits a solution which is the first hitting time of the joint process  $(W_t, t)$  of a *Root's barrier*:

**DEFINITION 1.2** (Root's barrier). A closed subset  $B$  of  $[-\infty, +\infty] \times [0, +\infty]$  is a *Root's barrier* if

- a).  $(x, +\infty) \in B$  if  $x \in [-\infty, +\infty]$ ;      b).  $(\pm\infty, t) \in B$  if  $t \in [0, \infty]$ ;
- c). if  $(x, t) \in B$ , then  $(x, s) \in B$  whenever  $s > t$ .

There have been a number of important contributions concerning Root's barriers (given that [potential ordering](#) holds). An immediately subsequent paper [Loynes \[1970\]](#) proved some elementary analytical properties of Root's barriers. Moreover, by posing the definition *regular barrier*, the uniqueness of Root's embedding was given in this paper.

Another important paper regarding Root's construction is [Röst \[1976\]](#) which vastly extends the generality of Root's existence result. More importantly, Röst firstly proved the optimality of Root's embedding, which was conjectured by [Kiefer \[1972\]](#), in the sense of *minimal residual expectation* (m.r.e., for short):

Amongst all solutions of [SEP](#)( $X, \nu, \mu$ ), the Root's solution  
minimises  $\mathbb{E}^\nu[(\tau - t)^+]$  simultaneously for all  $t > 0$ . (m.r.e.)

In [Cox and Wang \[2013\]](#), given [potential ordering](#), we derived the construction of Root's embeddings using variational inequalities. We also proposed the conjecture that, by slightly changing the terminal condition in our variational inequality, this construction method could be extended to the cases where [potential ordering](#) fails ([Cox and Wang \[2013, Rmk. 4.5\]](#)). In the same paper, an alternative proof of the m.r.e. property was given, which has an important application to the construction of

sub-hedging strategies in the financial context. Later, [Gassiat et al. \[2015\]](#) describes Root's barrier in terms of viscosity solutions of obstacle problems, and gives a rigorous proof of the existence of Root's barrier given [potential ordering](#).

In this work, we will extend the generality of the variational inequality construction of Root's embedding to the cases without [potential ordering](#). On the other hand, thanks to the rich results given in [Cox and Hobson \[2006\]](#) and [Cox \[2008\]](#), it will turn out that we can characterize minimal stopping times by the local times of the corresponding stopped process  $(\mathbb{E}^\nu[L_\tau^x])$ . This characterization then ensure that we can construct a minimal Root's embedding via a variational inequality with properly choosed terminal condition.

The paper will therefore proceed as follows: in [Section 2](#), we review some early results about Root's barriers. In [Section 3](#), the existence result and the construction of Root's barrier for general staring and target distributions are given. In [Section 4](#), we study the potentials of the corresponding stopped process (and their limit), and obtain a necessary and sufficient condition for a Root's stopping time to be minimal. At last, in [Section 5](#), we consider the optimality of non-UI Root's embeddings in the sense of *maximal principal expectation*, which can be regarded as the generalisation of minimal residual expectation ([m.r.e.](#)).

## 2 Preliminaries: Root's barriers for regular cases

We firstly review the previous results regarding Root's embeddings, which are useful throughout this work.

It was shown in [Loynes \[1970, Prop. 3\]](#) that the set  $B$  defined in [Definition 1.2](#) can be represented as a closed set bounded below by a lower semi-continuous function  $R : \mathbb{R} \rightarrow [0, +\infty]$ , i.e.

$$B = \{(x, t) : t \geq R(x)\}.$$

This representation has been helpful in the characterization of the law of the stopped process  $X^\tau$ . In the rest of this paper, we will say that a barrier is either a closed set described in [Definition 1.2](#), or equivalently its complement:

$$D = \{(x, t) : 0 < t < R(x)\} = (\mathbb{R} \times \mathbb{R}_+) \setminus B. \quad (2.1)$$

The corresponding stopping time is denoted by

$$\tau_D := \inf \{ t > 0 : (X_t, t) \notin D \} = \inf \{ t > 0 : t \geq R(X_t) \}.$$

Moreover, [Loynes \[1970, Prop. 1\]](#) says that,

for a Root's stopping time  $\tau_D$ , either  $\mathbb{P}[\tau_D < \infty] = 1$  or  $\mathbb{P}[\tau_D = \infty] = 1$ .

Throughout this work, we only consider the non-trivial case  $\mathbb{P}[\tau_D < \infty] = 1$ .

In [Cox and Wang \[2013\]](#), we consider the construction of Root's embeddings for time-homogeneous diffusions. Some constraints have been imposed on the underlying process  $X$ : We assume that

$$dX_t = \sigma(X_t)dW_t; \quad X_0 \sim \nu, \quad (2.2)$$

where  $\sigma : \mathbb{R} \rightarrow \mathbb{R}_+$  satisfies the usual condition: for some constant  $K > 0$ ,

$$|\sigma(x) - \sigma(y)| \leq K|x - y|; \quad 0 < \sigma^2(x) < K(1 + x^2); \quad \sigma \text{ is smooth on } \mathbb{R}. \quad (2.3)$$

In some circumstances, we need an additional condition:

$$1/K < \sigma(x) < K. \quad (2.4)$$

The following properties are given in [Cox and Wang \[2013\]](#), which enable us characterize the behaviour of the path of corresponding stopped process.

**PROPOSITION 2.1.** *Suppose  $X$  is a continuous process. Given a Root's barrier  $D$  and the corresponding stopping time is denoted by  $\tau_D$ , then we have,*

- a). *if  $(x, t) \in D$ ,  $\mathbb{P}^\nu[X_{t \wedge \tau_D} \in dx] = \mathbb{P}^\nu[X_t \in dx, t < \tau_D]$ ;*
- b). *if  $(x, t) \notin D$ , the path of  $X$  does not cross the horizontal level  $\{x\} \times (0, \infty)$  within the time interval  $(t \wedge \tau_D, \tau_D)$ .*

A direct corollary of [Proposition 2.1\(b\)](#) is that if  $(x, t) \notin D$ , then

$$L_{t \wedge \tau_D}^x = L_{\tau_D}^x, \quad \mathbb{P}^\nu\text{-a.s..}$$

These properties are local properties and do not rely on the uniform integrability condition, so they remain true for any Root's stopping time even if it is not a UI stopping time.

Applying [Proposition 2.1](#), we can connect Root's embeddings to specified free boundary problems. Denote the potential of the stopped process by

$$u(x, t) = -\mathbb{E}^\nu|x - X_{t \wedge \tau_D}|,$$

then  $u$  is of the class  $\mathcal{C}^0(\mathbb{R} \times \mathbb{R}_+) \cap \mathcal{C}^{2,1}(D)$ , and satisfies

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} \quad \text{on } D; \quad u(\cdot, 0) = U_\nu \quad \text{on } \mathbb{R}. \quad (2.5a)$$

Moreover, if  $\tau_D$  is a UI stopping time such that  $X_{\tau_D} \sim \mu$ , then

$$u(x, t) = U_\mu(x), \quad \text{if } (x, t) \notin D; \quad u(x, t) \longrightarrow U_\mu(x), \quad \text{as } t \rightarrow \infty. \quad (2.5b)$$

Note that the UI condition implies that  $U_\nu \geq U_\mu$ .

Cox and Wang [2013, Thm. 4.2] gives a one-to-one correspondence between Root's stopping and some analytic formulation. Before this result, we review the definition of variational inequality and its strong solution.

Suppose that  $\sigma$ ,  $u_0$ ,  $\bar{u}$  are functions defined on  $\mathbb{R}$  such that  $u_0 \geq \bar{u}$ , we denote the following variational inequality by  $\text{VI}(\sigma, u_0, \bar{u})$ ,

$$\begin{aligned} \min \left\{ \frac{\partial u}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}, \quad u - \bar{u} \right\} &= 0, & \text{on } \mathbb{R} \times \mathbb{R}_+; \\ u_0 - u(\cdot, 0) &= 0, & \text{on } \mathbb{R}. \end{aligned} \quad (\text{VI})$$

Then  $u$  is a *strong solution* to  $\text{VI}(\sigma, u_0, \bar{u})$  if for all  $\lambda$ ,  $T > 0$ ,

$$u \in L^\infty(0, T; H^{1, \lambda}), \quad \partial u / \partial t \in L^2(0, T; H^{0, \lambda}); \quad (2.6a)$$

$$\int_{\mathbb{R}} e^{-2\lambda|x|} (w - u) \left[ \frac{\partial u}{\partial t} dx - \frac{\sigma^2}{2} d \left( \frac{\partial u}{\partial x} \right) \right] \geq 0, \quad \text{a.e. } t \in (0, T), \quad (2.6b)$$

$$\text{for all } w \in H^{1, \lambda} \text{ such that } w \geq \bar{u}, \quad \text{a.e. } x \in \mathbb{R};$$

$$u(x, t) \geq \bar{u}(x), \quad u(x, 0) = u_0(x), \quad \text{a.e. } (x, t) \in \mathbb{R} \times \mathbb{R}_+. \quad (2.6c)$$

According to Bensoussan and Lions [1982, Thm. 2.2 & Sec. 2.15], under suitable assumption on the parameters,  $\text{VI}(\sigma, u_0, \bar{u})$  admits a unique strong solution. See also Cox and Wang [2013, Thm. 4.1].

Then the equivalence between Root's embeddings and variational inequalities is given by the following theorem.

**THEOREM 2.2** (Cox and Wang [2013]). *Suppose (2.2)-(2.4) hold. Let  $D$  be a Root's barrier such that the first hitting time  $\tau_D$  is a UI embedding for  $\text{SEP}(X, \nu, \mu)$ . Also, let  $u(x, t)$  be the unique strong solution to the variational inequality  $\text{VI}(\sigma, U_\nu, U_\mu)$ . Then we have*

$$u(x, t) = -\mathbb{E}^\nu |x - X_{t \wedge \tau_D}| \quad \text{and} \quad D = \{(x, t) : u(x, t) > U_\mu(x)\}.$$

As stated in Section 1,  $\tau_D$  is non-UI when potential ordering fails, and then (2.5b) does not hold any longer. On the other hand, since Proposition 2.1 does not rely on the UI assumption, one can find that (2.5a) still holds. Therefore, when potential ordering fails, in order to construct Root's embedding using variational inequalities, we only need to find a more general version of (2.5b). We will deal with this problem in next section, .

### 3 Existence and construction of Root's embeddings

Given  $U_\mu \leq U_\nu$ , there exists a Root's solution  $\tau_D$  for  $\text{SEP}(X, \nu, \mu)$  if  $U_\mu \leq U_\nu$ , and we can construct this solution via a variational inequality (see Section 1). However, when

potential ordering fails, we can't even be sure if the Root's embedding exists. From now on, we are concerned with the existence and construction of Root's embedding in general cases.

We always assume that  $X$  is a time-homogeneous diffusion given by (2.2) where the diffusion coefficient  $\sigma$  satisfies (2.3) and (2.4), and  $\nu, \mu$  are integrable distributions. Now we consider the variational inequality  $\text{VI}(\sigma, u_0, \bar{u})$  where the coefficients are defined as below,

$$\begin{aligned} \sigma & \text{ is the diffusion coefficient of } X \text{ and satisfies (2.3) and (2.4);} \\ u_0 &= U_\nu, \quad \bar{u} = U_\mu - C, \quad \text{where } C \geq 0 \text{ is a constant s.t. } u_0 \geq \bar{u}. \end{aligned} \quad (3.1)$$

As mentioned in Section 1, this variational inequality admits a unique strong solution. Denoting this strong solution by  $u$ , we define

$$D = \{(x, t) \in \mathbb{R} \times \mathbb{R}_+ : u(x, t) > \bar{u}(x)\}. \quad (3.2)$$

Obviously,  $D$  is an open set since  $u$  and  $\bar{u}$  is continuous.

In this section, we will see that  $D$  is a Root's barrier such that the first hitting time  $\tau_D = \inf \{t \geq 0 : (X_t, t) \notin D\}$  is a solution for  $\text{SEP}(X, \nu, \mu)$ .

The key observation is that one can connect the variational inequality to a particular optimal stopping problem.

As noted in Bensoussan and Lions [1982, Sec. 4.9, Chp. 3], the solution to  $\text{VI}(\sigma, u_0, \bar{u})$  is also the function which arises from solving the problem:

$$u(x, t) = \sup_{\theta \leq t} J_{x,t}(\theta), \quad \text{where } J_{x,t}(\theta) = \mathbb{E}^x [\bar{u}(X_\theta) \mathbb{1}_{\theta < t} + u_0(X_\theta) \mathbb{1}_{\theta = t}]. \quad (3.3)$$

Moreover, according to Cox and Wang [2013, Rmk 4.4], the solution to (3.3) is also given by

$$u(x, t) = J_{x,t}(\theta_t), \quad \text{where } \theta_t = \inf \{r \geq 0 : (X_r, t - r) \notin D\} \wedge t. \quad (3.4)$$

Using this result we firstly verify that the open set  $D$  is a Root's barrier.

**LEMMA 3.1.** *Let  $u$  be the strong solution to  $\text{VI}(\sigma, u_0, \bar{u})$  with coefficients given by (3.1). Then  $D$  defined in (3.2) is a Root's barrier.*

*Proof.* Using (3.4) and the strong Markov property of  $X$ , we have that

$$\begin{aligned} u(x, t) &= \mathbb{E}^x \left[ \bar{u}(X_{\theta_t}) \mathbb{1}_{\theta_t < t-s} + \bar{u}(X_{\theta_t}) \mathbb{1}_{t-s \leq \theta_t < t} + u_0(X_{\theta_t}) \mathbb{1}_{\theta_t = t} \right] \\ &= \mathbb{E}^x [\bar{u}(X_{\theta_t}) \mathbb{1}_{\theta_t < t-s} + u(X_{t-s}, s) \mathbb{1}_{\theta_t \geq t-s}] \\ &\leq \mathbb{E}^x [u(X_{\theta_t}, s) \mathbb{1}_{\theta_t < t-s} + u(X_{t-s}, s) \mathbb{1}_{\theta_t \geq t-s}] \\ &= \mathbb{E}^x [u(X_{(t-s) \wedge \theta_t}, s)] \\ &\leq u(x, s), \end{aligned} \quad (3.5)$$

where the last inequality follows from Jensen's inequality since  $u(\cdot, s)$  is concave. Thus,  $u(x, s) \geq u(x, t) > \bar{u}(x)$  whenever  $u(x, t) > \bar{u}(x)$  and  $s \leq t$ . It follows that  $D$  is a Root's barrier.  $\square$

**REMARK 3.2.** The monotonicity of  $u(\cdot, t)$  also can be found in [Gassiat et al. \[2015, Cor. 1\]](#), and they proved the result using PDE theory. The proof we present here is independently derived via the connection between optimal stopping problems and variational inequalities.

We have seen that the region  $D$  given by the strong solution  $u$  is a Root's barrier. When the corresponding hitting time  $\tau_D$  is a UI stopping time, according to [Theorem 2.2](#),  $u(x, t) = -\mathbb{E}[x - X_{t \wedge \tau_D}]$ . But what if the UI assumption fails? Next we will interpret the strong solution  $u(x, t)$  in a probabilistic viewpoint.

Since  $u$  satisfies (2.6b), for any testing function  $\phi \geq 0$ , we have

$$\int_{\mathbb{R}} e^{-2\lambda|x|} \phi(x) \left[ \frac{\partial u}{\partial t} dx - \frac{\sigma^2(x)}{2} d \left( \frac{\partial^2 u}{\partial x^2} \right) \right] \geq 0.$$

Then, since  $u$  is decreasing in  $t$  as shown in [Lemma 3.1](#),  $u$  is concave in  $x$ .

In addition, noting that  $|(u_0)'_-| \leq 1$  and the Radon measure defined by  $u_0''$  is  $-2\nu(dx)$ , we have, by (3.3) and Itô-Tanaka formula,

$$\begin{aligned} 0 &\leq u_0(x) - u(x, t) \leq u_0(x) - J_{x,t}(t) = u_0(x) - \mathbb{E}^x[u_0(X_t)] \\ &= -\mathbb{E}^x \left[ \int_0^t (u_0)'_-(X_s) dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a u_0''(da) \right] = \int_{\mathbb{R}} \mathbb{E}^x[L_t^a] \nu(da). \end{aligned}$$

Denote the transition density of  $X$  by  $p_t^X(x, y)$ , then we have (see e.g. [Wang \[2011, Prop. 2.5.2\]](#)),

$$\mathbb{E}^x[L_t^a] = \int_0^t \sigma^2(a) p_s^X(x, a) ds = \int_0^t \sigma^2(x) p_s^X(a, x) ds = \mathbb{E}^a[L_t^x].$$

It then follows from [Chacon \[1977, Lem. 2.2\]](#) that, as  $|x| \rightarrow \infty$ ,

$$\begin{aligned} 0 &\leq u_0(x) - u(x, t) \leq \int_{\mathbb{R}} \mathbb{E}^a[L_t^x] \nu(da) \leq \mathbb{E}^\nu[L_t^x] \\ &= \mathbb{E}^\nu|x - X_t| - \mathbb{E}^\nu|x - X_0| \longrightarrow 0. \end{aligned}$$

Thus, we can conclude that there exists some probability distribution, denoted by  $\mu_t$ , such that (see e.g. [Wang \[2011, Lem. 2.3.1\]](#))

$$u(\cdot, t) = U_{\mu_t} \leq U_\nu.$$



Moreover, since  $u(x, t)$  is decreasing in  $t$  and bounded below by  $\bar{u}(x)$ , we have that  $\lim_{t \rightarrow \infty} u(x, t)$  exists everywhere, and then, according to Chacon [1977, Sec. 2], there exist some constant  $C_L \geq 0$  and some measure  $\hat{\mu}$  defined on  $\mathbb{R}$  such that

$$\mu_t \Rightarrow \hat{\mu} \quad \text{and} \quad \hat{u}(x) := \lim_{t \rightarrow \infty} u(x, t) = U_{\hat{\mu}}(x) - C_L, \quad \text{for all } x \in \mathbb{R}, \quad (3.6)$$

so  $\hat{u}$  is uniformly continuous with modulus of continuity bounded by  $\hat{\mu}(\mathbb{R})$ . Then, by the continuity and the monotonicity of  $u$ , we have that

$$\hat{D} := \{(x, t) \in \mathbb{R} \times \mathbb{R}_+ : u(x, t) > \hat{u}(x)\} \quad \text{is a Root's barrier.} \quad (3.7)$$

and obviously  $\hat{D} \subseteq D$ . Denoting the corresponding hitting time of  $\hat{D}$  by  $\hat{\tau}$ , we then have that  $\hat{\tau} \leq \tau_D$ , and the following result.

**LEMMA 3.3.** *Denote the hitting time  $\hat{\tau} = \inf \{t > 0 : (X_t, t) \notin \hat{D}\}$ , then for all  $t > 0$ ,  $X_{\hat{\tau} \wedge t} \sim \mu_t$ , i.e.  $u(\cdot, t)$  is the potential of the distribution  $\mathcal{L}(X_{\hat{\tau} \wedge t})$ .*

*Proof.* Fix  $t > 0$ . For all  $s \geq 0$ , define  $v(x, s) = u(x, s \wedge t)$ . Then one can easily check that  $v(x, s)$  is the strong solution of  $\text{VI}(\sigma, U_\nu, u(\cdot, t))$ . In addition,  $u(\cdot, t) = U_{\mu_t} \leq U_\nu$  for some probability distribution  $\mu_t$ . Then, by Theorem 2.2, the hitting time  $\inf \{s > 0 : (W_s, s) \notin \tilde{D}_t\}$  is a UI embedding for  $\text{SEP}(X, \nu, \mu_t)$ , where

$$\tilde{D}_t := \{(x, s) : v(x, s) > u(x, t)\} = \{(x, s) : u(x, s \wedge t) > u(x, t)\}.$$

Then, to see  $X_{t \wedge \hat{\tau}} \sim \mu_t$ , we only need to show that  $\tilde{D}_t = \hat{D}_t$  where

$$\hat{D}_t := \{(x, s) \in \hat{D} : s < t\} = \{(x, s) : u(x, s) > \hat{u}(x), s < t\}.$$

For any  $(x, s) \in \tilde{D}_t$ , obviously  $s < t$ . Moreover,  $u(x, s) > u(x, t) \geq \hat{u}(x)$ , then  $(x, s) \in \hat{D}$ . Therefore,  $\tilde{D}_t \subset \hat{D}_t$ .

Conversely, for  $(x, s) \in \hat{D}_t$ , it follows directly that  $(x, s) \in \tilde{D}_t$  holds if  $u(x, t) = \hat{u}(x)$ . Assume that  $(x, s) \in \hat{D}_t$  and  $u(x, t) > \hat{u}(x)$ . Because  $u$  is continuous, there exists some  $T > t$  such that  $u(x, s) \geq u(x, t) > u(x, T)$ , i.e.  $(x, s), (x, T) \in \tilde{D}_T$ . One can easily check that  $u(x, r)$  is strictly decreasing in  $r$  for  $(x, r) \in \tilde{D}_T$ . Thus  $u(x, s) > u(x, t)$ , i.e.  $(x, s) \in \tilde{D}_t$ . Therefore we reach the conclusion  $\hat{D}_t \subset \tilde{D}_t$ .  $\square$

**REMARK 3.4.** If  $\hat{\tau} < \infty$  almost surely, we have  $X_{\hat{\tau} \wedge t} \rightarrow X_{\hat{\tau}}$  almost surely. Then, by (3.6), an immediate result from Lemma 3.3 is that  $\hat{\mu} = \mathcal{L}(X_{\hat{\tau}})$

Recalling the representation of Root's barriers given by (2.1), we denote the barrier function of  $D$  and  $\hat{D}$  by  $R$  and  $\hat{R}$ , i.e.,

$$D = \{(x, t) : 0 < t < R(x)\} \quad \text{and} \quad \hat{D} = \{(x, t) : 0 < t < \hat{R}(x)\}.$$

Since  $\widehat{D} \subset D$ , we have that  $\widehat{R}(x) \leq R(x)$  everywhere on  $\mathbb{R}$ . For any  $x$  such that  $\widehat{R}(x) < R(x)$ , we must have  $R(x) = \infty$ . Otherwise, we could have

$$\widehat{u}(x) = u(x, r) > \bar{u}(x), \quad \text{for all } r \in [\widehat{R}(x), R(x)).$$

It follows from the continuity of  $u$  that  $u(x, R(x)) = \widehat{u}(x) > \bar{u}(x)$ . This violates the definition of  $D$ . Thus,

$$Q = \{x : \widehat{R}(x) < R(x)\} = \{x : \widehat{R}(x) < \infty, R(x) = \infty\}. \quad (3.8)$$

Moreover, we have the following result.

**LEMMA 3.5.** *Let  $\widehat{\mu}$  be the measure defined in (3.6), then  $\widehat{\mu}(Q) = 0$ .*

*Proof.* Given  $x \in Q$  and  $t > \widehat{R}(x)$ , we have that  $(x, t) \in D/\widehat{D}$ , and then  $\bar{u}(x) < u(x, t) = \widehat{u}(x)$ . By the continuity of  $\bar{u}$  and  $\widehat{u}$ , there exists  $\varepsilon^*$  such that  $\bar{u}(y) < \widehat{u}(y)$  for all  $y \in (x - \varepsilon^*, x + \varepsilon^*)$ . Thus,

$$\bar{u}(y) < \widehat{u}(y) = u(y, \widehat{R}(y)),$$

which implies  $\widehat{R}(y) < R(y)$ . Thus  $(x - \varepsilon^*, x + \varepsilon^*) \times (0, \infty) \subset D$  by (3.8).

Now pick  $t > s > \widehat{R}(x)$ . Since  $u(x, s) = u(x, t) = \widehat{u}(x)$ , we must have  $u(x, s) = \mathbb{E}^x [u(X_{(t-s) \wedge \theta_t}, s)]$  by (3.5). To enforce this equality, we must have either  $X_{(t-s) \wedge \theta_t} \equiv x$  or  $u(\cdot, s)$  is linear on  $\{X_{(t-s) \wedge \theta_t}(\omega) : \omega \in \Omega\}$ .

Since  $(x - \varepsilon^*, x + \varepsilon^*) \times (0, \infty) \subset D$  and  $\sigma \neq 0$ , we have

$$\mathbb{P}^x[(t-s) \wedge \theta_t = t-s] \geq \mathbb{P}^x[\sup_{r \leq t-s} |X_r - x| < \varepsilon^*] > 0,$$

so  $X_{(t-s) \wedge \theta_t}$  is impossible to be deterministic. Thus,  $u(\cdot, s)$  must be linear on the set  $\{X_{(t-s) \wedge \theta_t}(\omega) : \omega \in \Omega\}$ . And hence, at least,  $u(x, s)$  is linear on  $(x - \varepsilon^*, x + \varepsilon^*)$ . Applying Lemma 3.3, we then conclude that

$$\widehat{\mu}_t((x - \varepsilon^*, x + \varepsilon^*)) = \mathbb{P}^\nu(X_{\widehat{\tau} \wedge s} \in (x - \varepsilon^*, x + \varepsilon^*)) = 0.$$

Noting that the choice of  $\varepsilon^*$  is not dependent on  $t$  or  $s$  and  $\widehat{\mu}_s \Rightarrow \widehat{\mu}$  as  $s \rightarrow \infty$ , we have that  $\widehat{\mu}((x - \varepsilon^*, x + \varepsilon^*)) = 0$ , so  $\widehat{\mu}(dx) = 0$ , and then  $\widehat{\mu}(Q) = 0$ .  $\square$

With the help of this lemma, we now are in a position to prove the existence result of Root's barrier for general starting and target distributions  $\nu$  and  $\mu$ .

**THEOREM 3.6.** *Assume that  $\nu, \mu$  are integrable probability distributions and  $X$  is given by (2.2) with (2.3) and (2.4). Let  $u(x, t)$  be the strong solution to VI( $\sigma, u_0, \bar{u}$ ) as defined in (3.1), and  $D$  be the set defined in (3.2) such that  $\tau_D$  is non-trivial (i.e.  $\tau_D < \infty$  almost surely). Then  $\tau_D$  solves SEP( $X, \nu, \mu$ ).*

*Proof.* The notions  $\hat{u}$ ,  $\hat{\mu}$ ,  $\hat{D}$  are as defined in (3.6) and (3.7). Also, as before, let  $R$  and  $\hat{R}$  be the barrier functions of  $D$  and  $\hat{D}$  respectively. Then, by (3.8),

$$\begin{cases} F := \{x : R(x) < +\infty\} = \{x : R(x) = \hat{R}(x)\}, \\ \hat{F} := \{x : \hat{R}(x) < +\infty\} = F \cup Q. \end{cases} \quad (3.9)$$

The set  $F$  is non-empty because  $\tau_D$  is non-trivial. It directly follows that

$$\bar{u}(x) = u(x, R(x)) = u(x, \hat{R}(x)) = \hat{u}(x), \quad \text{for all } x \in F.$$

Thus,

$$U_\mu - C \leq U_{\hat{\mu}} - C_L \quad \text{on } \mathbb{R}; \quad U_\mu - C = U_{\hat{\mu}} - C_L \quad \text{on } F. \quad (3.10)$$

And hence, for any  $x \in F$  and  $\varepsilon \in \mathbb{R}$ ,  $U_\mu(x + \varepsilon) - U_\mu(x) \leq U_{\hat{\mu}}(x + \varepsilon) - U_{\hat{\mu}}(x)$ . Then the left and right derivatives of the potentials satisfy

$$(U_\mu)'_- \geq (U_{\hat{\mu}})'_- \quad \text{and} \quad (U_\mu)'_+ \leq (U_{\hat{\mu}})'_+ \quad \text{on } F.$$

It follows that

$$\mu((-\infty, x)) \leq \hat{\mu}((-\infty, x)) \leq \hat{\mu}((-\infty, x]) \leq \mu((-\infty, x]), \quad \forall x \in F,$$

and hence, for  $x \in F$ ,

$$\begin{aligned} \mu((-\infty, x)) &= \hat{\mu}((-\infty, x)), & \text{if } \mu(\{x\}) &= 0; \\ \mu(\{x\}) &\geq \hat{\mu}(\{x\}), & \text{if } \mu(\{x\}) &> 0. \end{aligned} \quad (3.11)$$

Suppose that there exists some  $x_0 \in F$  such that  $\mu(\{x_0\}) > \hat{\mu}(\{x_0\}) > 0$ . According to Lemma 3.5 and (3.9), it follows that

$$\mu(F) > \hat{\mu}(F) = \hat{\mu}(\hat{F}).$$

Because  $R(X_{\hat{\tau}}) \leq \hat{\tau} \leq \tau_D < \infty$  almost surely, we have

$$\hat{\mu}(\hat{F}) = \mathbb{P}^\nu[\hat{R}(X_{\hat{\tau}}) < \infty] \geq \mathbb{P}^\nu[\hat{\tau} < \infty] = 1.$$

Thus  $\mu(F) > \hat{\mu}(\hat{F}) = 1$ . This result violates that  $\mu$  is probability measure. Therefore, the inequality in (3.11) is in fact an equality, and hence  $\mu$  and  $\hat{\mu}$  agree on  $F$ . Furthermore, since  $\mu(F) = \hat{\mu}(F) = \hat{\mu}(\hat{F}) = 1$ ,  $\mu$  and  $\hat{\mu}$  in fact agree on  $\mathbb{R}$ , that is,  $X_{\hat{\tau}} \sim \mu$ .

At last, by the equality in (3.10), we have  $C = C_L$  since  $\mu = \hat{\mu}$ . Thus,  $\bar{u} = \hat{u}$  on  $\mathbb{R}$ . It directly follows that  $D = \hat{D}$  by their definition, and hence  $X_{\tau_D} = X_{\hat{\tau}} \sim \mu$ .  $\square$

In this section, we have shown that for any integrable distribution  $\nu$  and  $\mu$ , even if [potential ordering](#) fails, we can construct a Root's solution of  $\text{SEP}(X, \nu, \mu)$  via solving  $\text{VI}(\sigma, U_\nu, U_\mu - C)$ . It also turns out that there maybe exist infinitely many Root's embeddings for  $\text{SEP}(X, \nu, \mu)$ , dependent on different choice of  $C$  in  $\text{VI}(\sigma, U_\nu, U_\mu - C)$ .

For the cases where  $U_\mu \leq U_\nu$ , one may think that  $C = 0$  is the best choice because such Root's embeddings are UI stopping times. For the general cases where  $U_\mu \not\leq U_\nu$ , it is impossible to find UI embedding for  $\text{SEP}(X, \nu, \mu)$ . In such circumstances, we replace the uniform integrability by minimality as the criterion for choosing  $C$ . In next section, we will see that we can choose suitable  $C$  such that the corresponding Root's embedding is a minimal stopping time.

## 4 Minimality of Root's embeddings

In the previous section, we have obtained an analytical construction of Root's embeddings for the cases where [potential ordering](#) fails. As mentioned in [Section 1](#), we need the embeddings to be minimal in the sense of [Monroe \[1972a\]](#). In this section, we study the minimality of embeddings, and then, we will see how to choose suitable boundary condition in the variational inequalities such that the corresponding Root's embeddings are minimal.

To this end, we firstly recall the following result ([Cox \[2008, Thm. 17\]](#)), which connects the minimality of stopping times to potential functions.

**THEOREM 4.1.** *Let  $T$  be a solution of  $\text{SEP}(W, \nu, \mu)$  where  $\nu$  and  $\mu$  are integrable. Define*

$$\mathcal{A} = \{x \in [-\infty, +\infty] : \lim_{y \rightarrow x} (U_\mu - U_\nu)(y) = C^*\}, \quad (4.1)$$

$$\text{where } C^* := \sup_{x \in \mathbb{R}} \{U_\mu(x) - U_\nu(x)\},$$

$$a_+ = \sup \{x \in \overline{\mathbb{R}} : x \in \mathcal{A}\} \quad \text{and} \quad a_- = \inf \{x \in \overline{\mathbb{R}} : x \in \mathcal{A}\}. \quad (4.2)$$

Moreover, denote the first hitting times of the set  $\mathcal{A}$  and the horizontal level  $\gamma$  by  $H_{\mathcal{A}}$  and  $H_\gamma$  respectively. Then the following statements are equivalent:

(i).  $T$  is minimal;

(ii).  $T \leq H_{\mathcal{A}}$  and for all stopping times  $R \leq S \leq T$

$$\mathbb{E}^\nu[W_S | \mathcal{F}_R] \leq W_R \quad \text{on } \{W_0 \geq a_-\};$$

$$\mathbb{E}^\nu[W_S | \mathcal{F}_R] \geq W_R \quad \text{on } \{W_0 \leq a_+\};$$

(iii).  $T \leq H_{\mathcal{A}}$  and for all stopping times  $S \leq T$

$$\mathbb{E}^\nu[W_T | \mathcal{F}_S] \leq W_S \quad \text{on } \{W_0 \geq a_-\};$$

$$\mathbb{E}^\nu[W_T | \mathcal{F}_S] \geq W_S \quad \text{on } \{W_0 \leq a_+\};$$

(iv).  $T \leq H_A$  and for all  $\gamma > 0$ ,

$$\mathbb{E}^\nu[W_T; T > H_{-\gamma}, W_0 \geq a_-] \leq -\gamma \mathbb{P}^\nu[T > H_{-\gamma}, W_0 \geq a_-];$$

$$\mathbb{E}^\nu[W_T; T > H_{+\gamma}, W_0 \leq a_+] \geq +\gamma \mathbb{P}^\nu[T > H_{+\gamma}, W_0 \leq a_+];$$

(v).  $T \leq H_A$  and as  $\gamma \rightarrow \infty$ ,

$$\gamma \mathbb{P}^\nu[T > H_{-\gamma}, W_0 \geq a_-] \rightarrow 0;$$

$$\gamma \mathbb{P}^\nu[T > H_{+\gamma}, W_0 \leq a_+] \rightarrow 0.$$

Further, if there exists  $a \in \mathbb{R}$  such that  $\mathbb{P}^\nu[T \leq H_a] = 1$ , then  $T$  is minimal.

The original proof of [Theorem 4.1](#) does not rely on any properties of Brownian motion beyond the strong Markov property and the continuity of paths, so this result can be extended to any continuous strong Markov processes.

Now, let  $\tau$  be a solution of [SEP](#) $(X, \nu, \mu)$  (not necessarily be of Root's type), we denote the potential of the corresponding stopped process by

$$u(x, t) = -\mathbb{E}^\nu[x - X_{t \wedge \tau}].$$

We are interested in what will happen to  $u(x, t)$  as  $t \rightarrow \infty$ .

If  $\tau$  is a UI stopping time, we immediately have that  $\lim_{t \rightarrow \infty} u(x, t) = U_\mu$ . For the non-UI cases, we firstly review the examples mentioned in [Section 1](#).

**EXAMPLE 4.2.** For some  $a > 0$ ,  $H_a = \inf\{t > 0 : W_t = a\}$  is a non-UI solution for [SEP](#) $(W, \delta_0, \delta_a)$ . Let  $u(x, t) = -\mathbb{E}^{\delta_0}[x - W_{t \wedge H_a}]$ . One can compute for  $x < a$ ,

$$\begin{aligned} u(x, t) &= x - 2x \cdot \Phi\left(\frac{x}{\sqrt{t}}\right) + 2(x - 2a) \cdot \Phi\left(\frac{x - 2a}{\sqrt{t}}\right) \\ &\quad - 2\sqrt{t} \cdot \left[ \phi\left(\frac{x}{\sqrt{t}}\right) - \phi\left(\frac{x - 2a}{\sqrt{t}}\right) \right] \\ &\rightarrow x - 2x \cdot \Phi(0) + (2x - 4a) \cdot \Phi(0) = x - 2a = -|x - a| - a, \end{aligned}$$

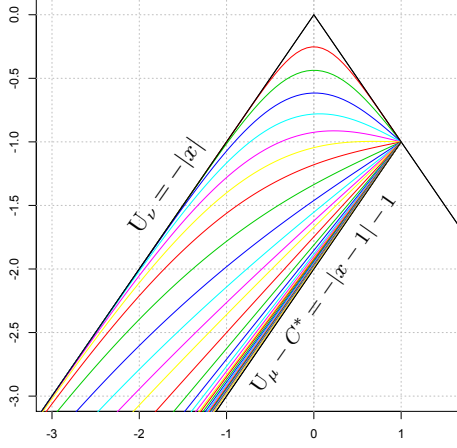
where  $\Phi$  and  $\phi$  denote the CDF and PDF of standard normal distribution respectively. For  $x \geq a$ , we have that  $u(x, t) = -x = -|x - a| - a$ . Therefore,  $\lim_{t \rightarrow \infty} u(x, t) = U_{\delta_a}(x) - a$  for all  $x \in \mathbb{R}$ .

**EXAMPLE 4.3.** For some  $a > 0$ ,  $H_0 = \inf\{t > 0 : W_t = 0\}$  is a non-UI solution for [SEP](#) $(W, (\delta_a + \delta_{-a})/2, \delta_0)$ . Then we have that

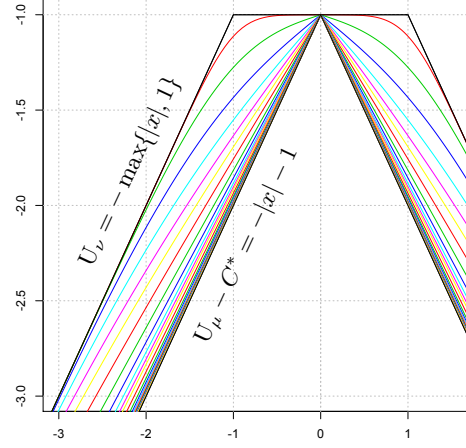
$$\begin{aligned} u(x, t) &= -(|x| + 2a) + (|x| + a) \cdot \Phi\left(\frac{|x| + a}{\sqrt{t}}\right) - (|x| - a) \cdot \Phi\left(\frac{|x| - a}{\sqrt{t}}\right) \\ &\quad + \sqrt{t} \cdot \left[ \phi\left(\frac{|x| + a}{\sqrt{t}}\right) - \phi\left(\frac{|x| - a}{\sqrt{t}}\right) \right]. \end{aligned}$$

It is easy to verify that  $\lim_{t \rightarrow \infty} u(x, t) = -|x| - a = U_{\delta_0}(x) - a$ .

Figure 1: The potential evolution described in Example 4.2 and Example 4.3 (Set  $a = 1$ ).



Example 4.2:  $\nu = \delta_0$ ,  $\mu = \delta_1$ .



Example 4.3:  $\nu = \frac{\delta_1 + \delta_{-1}}{2}$ ,  $\mu = \delta_0$ .

By the last line of Theorem 4.1, both embeddings given in the above examples are minimal. Denote the starting and target distributions by  $\nu$  and  $\mu$  respectively in these examples, one then can find that (see Figure 1)

$$\lim_{t \rightarrow \infty} u(x, t) = U_\mu(x) - C^*, \quad \text{where } C^* = \sup_{x \in \mathbb{R}} \{U_\mu(x) - U_\nu(x)\}.$$

This result can be extended to general cases as the following lemma.

**LEMMA 4.4.** *Given a diffusion process  $X$  satisfying (2.2)-(2.4), and integrable distributions  $\nu$  and  $\mu$ . Let  $\tau$  be a solution of SEP( $X, \nu, \mu$ ) such that  $\tau < \infty$  almost surely, and  $C^* := \sup_{x \in \mathbb{R}} \{U_\mu(x) - U_\nu(x)\}$ . Then*

$$\begin{aligned} \lim_{t \rightarrow \infty} u(x, t) &= U_\mu(x) - C_L, \quad \text{for all } x \in \mathbb{R}, \\ \text{where } C_L &= C^* + \inf_{x \in \mathbb{R}} \mathbb{E}^\nu [L_\tau^x]. \end{aligned} \tag{4.3}$$

In particular,  $C^* = C_L$  if  $\tau$  is a minimal stopping time.

*Proof.* Since  $t \wedge \tau \rightarrow \tau$  almost surely,  $X_{t \wedge \tau} \rightarrow X_\tau$  almost surely, and then  $\mathcal{L}(X_{t \wedge \tau}) \Rightarrow \mathcal{L}(X_\tau)$ . By Chacon [1977, Lem. 2.5], there exists a constant  $C_L$  such that

$$\lim_{t \rightarrow \infty} u(x, t) = U_\mu(x) - C_L, \quad \text{for all } x \in \mathbb{R}.$$

By martingale property and Tanaka's formula, we have that

$$\begin{aligned} -\infty < U_\mu(x) - u(x, t) &= \mathbb{E}^\nu \left[ \int_{t \wedge \tau}^\tau \text{sgn}(x - X_s) dX_s + (L_{t \wedge \tau}^x - L_\tau^x) \right] \\ &= \mathbb{E}^\nu \left[ \int_0^\tau \text{sgn}(x - X_s) dX_s + (L_{t \wedge \tau}^x - L_\tau^x) \right]. \end{aligned}$$

Then, by the monotone convergence theorem,

$$C_L \equiv U_\mu(x) - \lim_{t \rightarrow \infty} u(x, t) = \mathbb{E}^\nu \left[ \int_0^\tau \text{sgn}(x - X_s) dX_s \right], \quad \forall x \in \mathbb{R}.$$

It follows that, by the definition of  $C^*$ ,

$$\begin{aligned} C^* &= \sup_{x \in \mathbb{R}} \{U_\mu(x) - U_\nu(x)\} = \sup_{x \in \mathbb{R}} \mathbb{E}^\nu [|x - X_0| - |x - X_\tau|] \\ &= \sup_{x \in \mathbb{R}} \mathbb{E}^\nu \left[ \int_0^\tau \text{sgn}(x - X_s) dX_s - L_\tau^x \right] = C_L - \inf_{x \in \mathbb{R}} \mathbb{E}^\nu [L_\tau^x]. \end{aligned}$$

Now we assume additionally that  $\tau$  is a minimal stopping time. Recall the set  $\mathcal{A}$  defined in (4.1):

$$\mathcal{A} = \{x \in [-\infty, +\infty] : \lim_{y \rightarrow x} (U_\mu - U_\nu)(y) = C^*\}.$$

If  $\mathcal{A} \cap \mathbb{R} \neq \emptyset$ . Pick  $y \in \mathcal{A} \cap \mathbb{R}$ . Since  $\tau$  is minimal, by Theorem 4.1, we have that  $\tau \leq H_{\mathcal{A}} \leq H_y < \infty$ ,  $\mathbb{P}^\nu$ -a.s.. It follows that  $\mathbb{E}^\nu [L_\tau^y] \leq \mathbb{E}^\nu [L_{H_y}^y] = 0$ . Therefore,  $\inf_{x \in \mathbb{R}} \mathbb{E}^\nu [L_\tau^x] = 0$ .

If  $\mathcal{A} \cap \mathbb{R} = \emptyset$ , then  $\mathcal{A} \subset \{-\infty, +\infty\}$ . Without loss of generality, we assume that  $+\infty \in \mathcal{A}$ . For any  $y \in \mathbb{R}$ , denoting  $a^+ := \max(a, 0) = (|a| + a)/2$ , then

$$\begin{aligned} \mathbb{E}^\nu [L_{t \wedge \tau}^y] &= \mathbb{E}^\nu |X_{t \wedge \tau} - y| - \mathbb{E}^\nu |X_0 - y| \\ &= 2\mathbb{E}^\nu [(X_{t \wedge \tau} - y)^+] - 2\mathbb{E}^\nu [(X_0 - y)^+]. \end{aligned} \tag{4.4}$$

Since  $\tau$  is a minimal stopping time, by Theorem 4.1 (iii),

$$(X_{t \wedge \tau} - y)^+ \leq \left( \mathbb{E}^\nu [X_\tau - y | \mathcal{F}_{t \wedge \tau}] \right)^+ \leq \mathbb{E}^\nu [(X_\tau - y)^+ | \mathcal{F}_{t \wedge \tau}].$$

Since  $\mu$  is an integrable distribution, the process  $\{(X_{t \wedge \tau} - y)^+\}$  is uniformly integrable. Now letting  $t$  go to infinity in (4.4), we have that

$$\begin{aligned} \mathbb{E}^\nu [L_\tau^y] &= 2\mathbb{E}^\nu [(X_\tau - y)^+] - 2\mathbb{E}^\nu [(X_0 - y)^+] \\ &= [U_\nu(y) - U_\mu(y)] + (m_\mu - m_\nu), \end{aligned}$$

where  $m_\nu$  and  $m_\mu$  denote the mean values of  $\nu$  and  $\mu$  respectively. By Chacon [1977, Lem. 2.2], we have

$$\lim_{y \rightarrow +\infty} [U_\nu(y) + (y - m_\nu)] = \lim_{y \rightarrow +\infty} [U_\mu(y) + (y - m_\mu)] = 0.$$

Thus,  $\lim_{y \rightarrow +\infty} [U_\mu(y) - U_\nu(y)] = m_\mu - m_\nu$ . Therefore, we have that  $\mathbb{E}^\nu [L_\tau^y] \rightarrow 0$  as  $y \rightarrow +\infty$ , and then  $\inf_{y \in \mathbb{R}} \mathbb{E}^\nu [L_\tau^y] = 0$ . The case in which  $-\infty \in \mathcal{A}$  is similar.  $\square$

We have seen that  $C^* = C_L$  (or equivalently,  $\inf_{x \in \mathbb{R}} \mathbb{E}^\nu[L_\tau^x] = 0$ ) is the necessary condition for the minimality. However, our aim in this section is to show that the Root's embedding given by  $\text{VI}(\sigma, U_\nu, U_\mu - C^*)$  is minimal. To this end, next we will see  $C^* = C_L$  is also a sufficient condition.

**THEOREM 4.5.** *Under the same assumptions imposed in Lemma 4.4,  $\tau$  is a minimal stopping if and only if  $\lim_{t \rightarrow \infty} u(x, t) = U_\mu(x) - C^*$  for all  $x \in \mathbb{R}$ , or equivalently,  $\inf_{x \in \mathbb{R}} \mathbb{E}^\nu[L_\tau^x] = 0$ .*

*Proof.* It has been shown in Lemma 4.4 that  $\lim_{t \rightarrow \infty} u(x, t) = U_\mu(x) - C^*$  if  $\tau$  is minimal. It only remains to show the "if" part. Now we suppose that  $\lim_{t \rightarrow \infty} u(x, t) = U_\mu(x) - C^*$ . Let  $\mathcal{A}$  be the set defined in (4.1).

If  $\mathcal{A} \cap \mathbb{R} \neq \emptyset$ , we can pick some  $y \in \mathcal{A} \cap \mathbb{R}$ . Since potential functions are continuous and  $u(x, t) \rightarrow U_\mu(x) - C^*$ , we have that  $\lim_{t \rightarrow \infty} u(y, t) = U_\nu(y)$ . Then by Tanaka's formula and monotone convergence theorem,

$$\mathbb{E}^\nu[L_\tau^y] = \lim_{t \rightarrow \infty} \mathbb{E}^\nu[L_{t \wedge \tau}^y] = U_\nu(y) - \lim_{t \rightarrow \infty} u(y, t) = 0,$$

and hence,  $L_\tau^y = 0$ ,  $\mathbb{P}^\nu$ -a.s.. Since  $X$  is a recurrent process,  $\tau \leq H_y$  almost surely, and  $\tau$  is a minimal stopping time by the last line of Theorem 4.1.

If  $\mathcal{A} = \{+\infty\}$ , then  $C^* = \lim_{y \rightarrow +\infty} [U_\mu(y) - U_\nu(y)] = m_\mu - m_\nu$  (see the proof of Lemma 4.4). Because  $u(x, t) \rightarrow U_\mu(x) - C^*$ , we have that

$$\begin{aligned} 2\mathbb{E}^\nu[X_\tau^+ - X_{t \wedge \tau}^+] &= \mathbb{E}^\nu[|X_\tau| - |X_{t \wedge \tau}|] + \mathbb{E}^\nu[X_\tau - X_{t \wedge \tau}] \\ &= [u(0, t) - U_\mu(0)] + (m_\mu - m_\nu) \\ &\rightarrow (m_\mu - m_\nu) - C^* = 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Then, by Scheffé's Lemma,  $X_{t \wedge \tau}^+$  converges to  $X_\tau^+$  in  $L^1$ , and  $\{X_{t \wedge \tau}^+\}$  is uniformly integrable. Therefore, as  $\gamma \rightarrow +\infty$

$$\begin{aligned} \gamma \mathbb{P}^\nu[\tau > H_\gamma] &= \gamma \mathbb{P}^\nu[\tau > H_\gamma, X_0 \geq \gamma] + \gamma \mathbb{P}^\nu[\tau > H_\gamma, X_0 < \gamma] \\ &\leq \gamma \cdot \nu([\gamma, \infty)) + \gamma \mathbb{P}^\nu[\tau > H_\gamma, X_0 < \gamma] \\ &\leq \gamma \cdot \nu([\gamma, \infty)) + \gamma \mathbb{P}^\nu[X_{H_\gamma \wedge \tau} \geq \gamma, X_0 < \gamma] \\ &\leq \mathbb{E}^\nu[X_0; X_0 \geq \gamma] + \mathbb{E}^\nu[X_{H_\gamma \wedge \tau}; X_{H_\gamma \wedge \tau} \geq \gamma] \rightarrow 0. \end{aligned}$$

Then it follows from Theorem 4.1 (v) that  $\tau$  is minimal. The case in which  $\mathcal{A} = \{-\infty\}$  is similar.

At last, if  $\mathcal{A} = \{-\infty, +\infty\}$ , we have that  $m_\mu - m_\nu = C^* = m_\nu - m_\mu$ , and then  $m_\mu = m_\nu$  and  $C^* = 0$ . Similar as above, one can find that the processes  $\{X_{t \wedge \tau}^\pm\}$  are uniformly integrable, so  $\{X_{t \wedge \tau}\}$  is uniformly integrable. Then  $\tau_D$  is minimal according to Monroe [1972a, Thm 3].  $\square$



Thanks to [Theorem 4.5](#), we can directly tell if the Root's embedding given by [Theorem 3.6](#) is minimal or not, and [Theorem 2.2](#) can be easily generalised as follows.

**THEOREM 4.6.** *Assume that  $\nu, \mu$  are integrable probability distributions and  $X$  is given by (2.2) with (2.3) and (2.4). Let  $u(x, t)$  be the strong solution to [VI](#)( $\sigma, U_\nu, U_\mu - C^*$ ) where  $C^* := \sup_{x \in \mathbb{R}} \{U_\mu(x) - U_\nu(x)\}$ , and  $D$  be the set defined in (3.2) such that  $\tau_D$  is non-trivial (i.e.  $\tau_D < \infty$  almost surely). Then  $\tau_D$  is a minimal embeddings of Root's type for [SEP](#)( $X, \nu, \mu$ ).*

## 5 Optimality of minimal Root's embeddings

As well-known, the UI embedding of Root's type is remarkable because it is of minimal residual expectation ([m.r.e.](#)). A natural question now arises: can we generalise this optimality result to non-UI Root's embeddings? When the stopped process  $X^\tau$  is not uniformly integrable, we cannot expect that  $\mathbb{E}^\nu[(\tau - t)^+]$  is finite. Thus, we study the quantity  $\mathbb{E}^\nu[\tau \wedge t] = \mathbb{E}^\nu[\tau - (\tau - t)^+]$  instead. We conjecture that the minimal Root's embedding  $\tau$  is of *maximal principal expectation*, that is,

*Amongst all minimal solutions of [SEP](#)( $X, \nu, \mu$ ),  
the Root's solution maximises  $\mathbb{E}^\nu[\tau \wedge t]$  simultaneously for all  $t > 0$ .*

If [potential ordering](#) holds, let  $\tau_D$  and  $\tau$  be UI solutions of [SEP](#)( $W, \nu, \mu$ ), where  $\tau_D$  is of Root's type. Then, since  $\tau_D$  is of [m.r.e.](#), we have that

$$\mathbb{E}^\nu[\tau_D \wedge t] = \mathbb{E}^\nu[\tau_D] - \mathbb{E}^\nu[(\tau_D - t)^+] \geq \mathbb{E}^\nu[\tau] - \mathbb{E}^\nu[(\tau - t)^+] = \mathbb{E}^\nu[\tau \wedge t].$$

For general cases in which [potential ordering](#) fails, we suppose that  $\tau$  is a minimal embedding for [SEP](#)( $X, \nu, \mu$ ), and the corresponding stopped potential is denoted by

$$u^\tau(x, t) = -\mathbb{E}^\nu|x - X_{t \wedge \tau}|.$$

One can easily check that  $u^\tau(x, t)$  is decreasing in  $t$ . According to [Theorem 4.5](#),  $u^\tau(x, t)$  converges to  $U_\mu(x) - C^*$  as  $t \rightarrow \infty$ , and hence  $u(x, t) \geq U_\mu(x) - C^*$  for all  $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ .

Since  $u^\tau(x, t) = -U_\nu(x) - \mathbb{E}^\nu[L_{t \wedge \tau}^x]$  for all  $t \geq 0$ , roughly we have that

$$\begin{aligned} u^\tau(x, t + \delta) - u^\tau(x, t) &= -\lim_{\varepsilon \downarrow 0} \mathbb{E}^\nu \left[ \frac{1}{2\varepsilon} \int_{t \wedge \tau}^{(t+\delta) \wedge \tau} \sigma^2(X_s) \mathbb{1}_{X_s \in (x-\varepsilon, x+\varepsilon)} ds \right] \\ &= -\lim_{\varepsilon \downarrow 0} \frac{\sigma^2(x)}{2\varepsilon} \int_t^{t+\delta} \mathbb{P}[X_s \in (x-\varepsilon, x+\varepsilon), s < \tau] ds. \end{aligned}$$

Since  $\mathbb{P}[X_s \in (x - \varepsilon, x + \varepsilon), s < \tau] \leq \mathbb{P}[X_{s \wedge \tau} \in (x - \varepsilon, x + \varepsilon)]$ , we have that

$$\begin{aligned} u^\tau(x, t + \delta) - u^\tau(x, t) &\geq -\lim_{\varepsilon \downarrow 0} \frac{\sigma^2(x)}{2\varepsilon} \int_t^{t+\delta} \mathbb{P}[X_{s \wedge \tau} \in (x - \varepsilon, x + \varepsilon)] ds \\ &= \frac{\sigma^2(x)}{2} \int_t^{t+\delta} \lim_{\varepsilon \downarrow 0} \frac{(u^\tau)'_-(x + \varepsilon) - (u^\tau)'_+(x - \varepsilon)}{2\varepsilon} ds \\ &= \frac{\sigma^2(x)}{2} \int_t^{t+\delta} \frac{\partial^2}{\partial x^2} u^\tau(x, s) ds. \end{aligned}$$

It follows that

$$\left( \frac{\partial}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} \right) u^\tau \geq 0, \quad \text{on } \mathbb{R} \times \mathbb{R}_+.$$

We have to mention that the argument above is just an intuitive illustration without technique details, and we shall refer readers to [Gassiat et al. \[2015, Thm. 1\]](#) for a rigorous proof of this result. Now we can conclude that  $u^\tau$  is a viscosity supersolution<sup>1</sup> of

$$\begin{aligned} \min \left\{ \frac{\partial u}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}, u - (U_\mu - C^*) \right\} &= 0, & \text{on } \mathbb{R} \times \mathbb{R}_+; \\ U_\nu - u(\cdot, 0) &= 0, & \text{on } \mathbb{R}. \end{aligned} \quad (5.1)$$

On the other hand, if  $\tau_D$  is a Root's embedding for  $\text{SEP}(X, \nu, \mu)$  and it is minimal, by [Theorem 4.6](#), one can find that  $u^{\tau_D}(x, t) = -\mathbb{E}^\nu|x - X_{t \wedge \tau_D}|$  is the viscosity solution of (5.1). Therefore, by the comparison theorem for the viscosity solutions (see e.g. [Gassiat et al. \[2015, Thm. 5\]](#)), we have the following result.

**PROPOSITION 5.1.** *Assume that  $\nu, \mu$  are integrable probability distributions and  $X$  is given by (2.2) with (2.3) and (2.4). Suppose  $\tau$  and  $\tau_D$  are two minimal embeddings for  $\text{SEP}(X, \nu, \mu)$  among which  $\tau_D$  is of Root's type. Then*

$$u^\tau(x, t) \geq u^{\tau_D}(x, t) \geq U_\mu(x) - C^*, \quad \text{for all } (x, t) \in \mathbb{R} \times \mathbb{R}_+. \quad (5.2)$$

According to this result, given any time level, the stopped distribution generated by Root's embedding is more "closed" to the target  $U_\mu - C^*$ .

The following result is an immediate application of [Proposition 5.1](#), and it is helpful when proving the optimality of minimal Root's embeddings.

**COROLLARY 5.2.** *Given the same assumption imposed in [Proposition 5.1](#), we have that, for any convex function  $Z$  such that  $\mathbb{E}^\nu[Z(X_{t \wedge \tau_D})] < \infty$ ,*

$$E^\nu[Z(X_{t \wedge \tau_D}) - Z(X_{t \wedge \tau})] \geq 0.$$

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<sup>1</sup>We refer the readers to [Gassiat et al. \[2015\]](#), [Fleming and Soner \[2006\]](#) for more details about viscosity solutions mentioned here.

*Proof.* Denote the distributions of  $X_{t \wedge \tau}$  and  $X_{t \wedge \tau_D}$  by  $\tilde{\nu}$  and  $\tilde{\mu}$  respectively, and an independent stochastic process with the same law as  $X$  by  $\tilde{X}$ . By (5.2), there exists a UI embedding for  $\text{SEP}(\tilde{X}, \tilde{\nu}, \tilde{\mu})$ , denoted by  $\theta$ . For any convex function  $Z$  such that  $\mathbb{E}^\nu[Z(X_{t \wedge \tau_D})] < \infty$ , we then have that

$$\begin{aligned} \mathbb{E}^\nu[Z(X_{t \wedge \tau_D})] &= \mathbb{E}^{\tilde{\nu}}[Z(\tilde{X}_\theta)] = \int_{\mathbb{R}} \mathbb{E}^x[Z(\tilde{X}_\theta)] \tilde{\nu}(dx) \\ &\geq \int_{\mathbb{R}} Z(x) \tilde{\nu}(dx) = \mathbb{E}^{\tilde{\nu}}[Z(\tilde{X}_0)] = \mathbb{E}^\nu[Z(X_{t \wedge \tau})]. \end{aligned}$$

□

Now we are at the position to consider the optimality result of minimal Root's embedding. Given a increasing *concave* function  $F$  with  $F(0) = 0$ , in the following work, we shall be able to give a pathwise inequality which encodes *the maximal principal expectation property* in the sense that we can find a supermartingale  $G_t$  and a function  $H(x)$  such that  $F(t) \leq G_t + H(X_t)$ , and such that the equality holds when  $t = \tau_D$ , and  $G_{t \wedge \tau_D}$  is a martingale. And then, by performing some limit arguments, we can show the minimal Root's embedding  $\tau_D$  maximizes  $\mathbb{E}[F(\tau)]$  among all minimal stopping time  $\tau$  such that  $X_\tau \sim \mu$ , and then  $\tau_D$  is of maximal principal expectation if we let  $F(s) = s \wedge t$ .

Assume  $D$  is the Root's barrier, with the barrier function  $R : \mathbb{R} \rightarrow \overline{\mathbb{R}}_+$ , such that  $\tau_D$  is a minimal embedding for  $\text{SEP}(X, \nu, \mu)$ . Denote that  $f(t) = F'_+(t)$  is the right derivative of  $F$ . We always assume that

$$\alpha := f(0) < \infty. \quad (5.3)$$

Similar as in our work Cox and Wang [2013, Sec. 5], we define

$$G(x, t) := \int_0^t M(x, s) ds - Z(x), \quad H(x) := \int_0^{R(x)} [f(s) - M(x, s)] ds + Z(x),$$

where

$$M(x, t) := \mathbb{E}^{(x, t)}[f(\tau_D)] \quad \text{and} \quad Z(x) := \int_0^x \int_0^y [2M(z, 0)/\sigma^2(z)] dz dy.$$

Then one can find the following results.

**PROPOSITION 5.3.** *Under the same assumption imposed in Proposition 5.1, assume that  $F$  is an increasing concave function such that (5.3) holds, then*

$$G(x, t) + H(x) \geq F(t), \quad \text{for all } (x, t) \in \mathbb{R}, \quad (5.4)$$

Moreover, define  $A(x) := \int_0^x \int_0^y \frac{2\alpha}{\sigma^2(z)} dz dy$ , and suppose that

$$\begin{cases} \mathbb{E}^\nu \left[ \int_0^T A'(X_s)^2 \sigma(X_s)^2 ds \right] < \infty, & \text{for all } T > 0. \\ \mathbb{E}^\nu[A(X_0)] < \infty. \end{cases} \quad (5.5)$$

Then

$$\begin{cases} \text{the process } \{G(X_{t \wedge \tau_D}, t \wedge \tau_D)\}_{t \geq 0} \text{ is a martingale,} \\ \text{the process } \{G(X_t, t)\}_{t \geq 0} \text{ is a supermartingale.} \end{cases} \quad (5.6)$$

One can prove this proposition using the same idea as in [Cox and Wang \[2013, Sec. 5\]](#). However, such a proof would be long. Here we use our results in [Cox and Wang \[2013\]](#) to prove this proposition directly.

*Proof.* Using (5.3), we can define, for all  $t \geq 0$ ,

$$\tilde{F}(t) := \alpha t - F(t) \quad \text{and} \quad \tilde{f}(t) := \tilde{F}'_+(t) = \alpha - f(t).$$

It is easy to check that  $\tilde{F}$  is an increasing convex function. Define

$$\begin{aligned} \tilde{G}(x, t) &:= \int_0^t \tilde{M}(x, s) ds - \tilde{Z}(x), \quad \tilde{H}(x) := \int_0^{R(x)} [\tilde{f}(s) - \tilde{M}(x, s)] ds + \tilde{Z}(x), \\ \text{where } \tilde{M}(x, t) &:= \mathbb{E}^{(x, t)} [\tilde{f}(\tau_D)] \quad \text{and} \quad \tilde{Z}(x) := \int_0^x \int_0^y \frac{2\tilde{M}(z, 0)}{\sigma^2(z)} dz dy. \end{aligned}$$

According to [Cox and Wang \[2013, Prop. 5.1\]](#),

$$\tilde{G}(x, t) + \tilde{H}(x) \leq \tilde{F}(t), \quad \text{for all } (x, t) \in \mathbb{R}, \quad (5.7)$$

Since  $\tilde{M}(x, t) = \alpha - M(x, t)$ , we have that  $\tilde{Z}(x) = A(x) - Z(x)$ , and

$$G(x, t) = [\alpha t - A(x)] - \tilde{G}(x, t), \quad H(x) = A(x) - \tilde{H}(x). \quad (5.8)$$

Thus, for all  $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ , by (5.7),

$$G(x, t) + H(x) = \alpha t - [\tilde{G}(x, t) + \tilde{H}(x)] \geq \alpha t - \tilde{F}(t) = F(t).$$

On the other hand, since  $\tilde{f}$  is bounded by 0 and  $\alpha$ , it follows from (5.5) that

$$\mathbb{E}^\nu \left[ \int_0^T \tilde{Z}'(X_s)^2 \sigma(X_s)^2 ds \right] < \infty, \quad \text{for all } T > 0; \quad \mathbb{E}^\nu [\tilde{Z}(X_0)] < \infty.$$

Then it follows from [Cox and Wang \[2013, Lem. 5.2\]](#)<sup>2</sup> that

$$\begin{cases} \text{the process } \{\tilde{G}(X_{t \wedge \tau_D}, t \wedge \tau_D)\}_{t \geq 0} \text{ is a martingale,} \\ \text{the process } \{\tilde{G}(X_t, t)\}_{t \geq 0} \text{ is a submartingale.} \end{cases} \quad (5.9)$$

Using Itô's lemma, one can easily check that  $\{\alpha t - A(X_t)\}$  is a martingale, then (5.6) follows from (5.8) and (5.9) directly.  $\square$

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<sup>2</sup> Note that we do not need the assumption that  $\tau_D$  is a UI stopping time in the proof of [Cox and Wang \[2013, Lem. 5.2\]](#).

Applying the pathwise inequality given in [Proposition 5.3](#), now we can show that the minimal Root's embedding is of the maximal principal expectation.

**THEOREM 5.4.** *Under the same assumption imposed in [Proposition 5.1](#), assume that  $F$  is an increasing concave function such that [\(5.3\)](#) and [\(5.5\)](#) hold, then  $\mathbb{E}^\nu[F(\tau_D)] \geq \mathbb{E}^\nu[F(\tau)]$ .*

*Proof.* We begin by considering the case where

$$f(t) = 0, \quad \text{for all } t > N. \quad (5.10)$$

Let  $\tau$  be a minimal stopping time such that  $X_\tau \sim \mu$ . By [\(5.6\)](#),

$$\mathbb{E}^\nu[G(X_{\tau_D \wedge t}, \tau_D \wedge t)] = \mathbb{E}^\nu[G(X_0, 0)] \geq \mathbb{E}^\nu[G(X_{\tau \wedge t}, \tau \wedge t)].$$

It follow that

$$\mathbb{E}^\nu \left[ \int_0^{t \wedge \tau_D} M(X_{t \wedge \tau_D}, s) ds - \int_0^{t \wedge \tau} M(X_{t \wedge \tau}, s) ds \right] \geq \mathbb{E}^\nu [Z(X_{t \wedge \tau_D}) - Z(X_{t \wedge \tau})].$$

Noting that  $Z$  is a convex function, and [\(5.5\)](#) guarantees the integrability of  $Z(X_{t \wedge \tau_D})$ , according to [Corollary 5.2](#), we have

$$\mathbb{E}^\nu \left[ \int_0^{t \wedge \tau_D} M(X_{t \wedge \tau_D}, s) ds - \int_0^{t \wedge \tau} M(X_{t \wedge \tau}, s) ds \right] \geq 0.$$

Because  $F$  satisfies [\(5.3\)](#) and [\(5.10\)](#), we have

$$\int_0^t M(x, s) ds = \int_0^{t \wedge N} \mathbb{E}^{(x, s)}[f(\tau_D)] ds \leq \alpha N < \infty.$$

Then, by dominated convergence theorem,

$$\mathbb{E}^\nu \left[ \int_0^{\tau_D} M(X_{\tau_D}, s) ds - \int_0^\tau M(X_\tau, s) ds \right] \geq 0.$$

Since  $\mathcal{L}(X_{\tau_D}) = \mathcal{L}(X_\tau)$ , it follows from [\(5.4\)](#) that

$$\begin{aligned} \mathbb{E}^\nu[F(\tau_D) - F(\tau)] &\geq \mathbb{E}^\nu[G(X_{\tau_D}, \tau_D) + H(X_{\tau_D})] - \mathbb{E}^\nu[G(X_\tau, \tau) + H(X_\tau)] \\ &= \mathbb{E}^\nu \left[ \int_0^{\tau_D} M(X_{\tau_D}, s) ds - \int_0^\tau M(X_\tau, s) ds \right] \geq 0. \end{aligned}$$

To observe that the result still holds when [\(5.10\)](#) does not hold, observe that [\(5.3\)](#), [\(5.5\)](#) and [\(5.10\)](#) hold for  $F_N(t) = F(t \wedge N)$ , and then we can apply the above argument to  $F_N(t)$  to get  $\mathbb{E}^\nu[F_N(\tau_D)] \geq \mathbb{E}^\nu[F_N(\tau)]$ . Then the conclusion follows from monotone convergence theorem that  $\mathbb{E}^\nu[F(\tau_D)] = \lim_{N \rightarrow \infty} \mathbb{E}^\nu[F(\tau_D \wedge N)] \geq \lim_{N \rightarrow \infty} \mathbb{E}^\nu[F(\tau \wedge N)] = \mathbb{E}^\nu[F(\tau)]$ .  $\square$

**REMARK 5.5.** One may find that the minimality is not so necessary in the optimality proof. In fact, only [Proposition 5.1](#) relies on the minimality.

Now we consider the case where  $\tau_D$  and  $\tau$  are solutions of  $\text{SEP}(W, \nu, \mu)$ , where  $\tau_D$  is of Root's type. Furthermore, we assume that

$$-\lim_{t \rightarrow \infty} \mathbb{E}^\nu |x - X_{t \wedge \tau}| = U_\mu(x) - C = -\lim_{t \rightarrow \infty} \mathbb{E}^\nu |x - X_{t \wedge \tau_D}|, \quad (5.11)$$

for some constant  $C \geq C^*$ . Then, similar as [Proposition 5.1](#), we have that

$$u^\tau(x, t) \geq u^{\tau_D}(x, t) \geq U_\mu(x) - C^*, \quad \text{for all } (x, t) \in \mathbb{R} \times \mathbb{R}_+.$$

Therefore, the result in [Corollary 5.2](#) still holds in this situation, and the maximal principal expectation property can be slightly generalised as follows

*Amongst all solutions of  $\text{SEP}(X, \nu, \mu)$  such that (5.11) holds,  
the Root's solution maximises  $\mathbb{E}^\nu[\tau \wedge t]$  simultaneously for all  $t > 0$ .*

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